On characterisation of Markov processes via martingale problems

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Abstract. It is well-known that well-posedness of a martingale problem in the class of continuous (or r.c.l.l.) solutions enables one to construct the associated transition probability functions. We extend this result to the case when the martingale problem is well-posed in the class of solutions which are continuous in probability. This extension is used to improve on a criterion for a probability measure to be invariant for the semigroup associated with the Markov process. We also give examples of martingale problems that are well-posed in the class of solutions which are continuous in probability but for which no r.c.l.l. solution exists.

Keywords. Martingale problem; Markov processes; semigroup; path properties.

1. Introduction

The seminal paper on multi-dimensional diffusions by Stroock and Varadhan [12] introduced martingale problems as a way of construction and study of Markov processes. Since then, this approach has been used successfully in several contexts such as interacting particle systems, Markov processes associated with Boltzmann equation, nonlinear filtering theory, controlled Markov processes, branching processes etc. A good account of the 'theory of martingale problems' is given in the book by Ethier and Kurtz [7]. To construct a Markov process, the martingale problem approach allows one to construct the process for each initial condition separately and a general result gives the measurability of the associated transition probability function. To proceed, we give the basic definitions here.

Given an operator A with domain $\mathscr{D}(A) \subseteq C_b(E)$ and range subset of $C_b(E)$ (where E is a complete separable metric space), a process X_t^x adapted to a filtration $(\mathscr{F}_t)$ is said to be a solution to the (A, δ_x) martingale problem if for all $f \in \mathscr{D}(A)$,

$$f(X_t^x) - \int_0^t Af(X_u^x)\mathrm{d}u \text{ is a } (\mathscr{F}_t)\text{-martingale} \qquad (1.1)$$

and

$$\mathbb{P}(X_0^x = x) = 1. \qquad (1.2)$$

The martingale problem for A is said to be well-posed in the class of r.c.l.l. solutions if for all x there exists a r.c.l.l. process (X_t^x) satisfying (1.1) and (1.2) and further for two such processes satisfying (1.1) and (1.2) (defined possibly on different probability

spaces), the finite dimensional distributions are the same. Well-posedness in the class of continuous solutions or measurable solutions is similarly defined. A well-known result, which has its origins in the work of Stroock and Varadhan [13] says that if the martingale problem for A is well-posed in the class of r.c.l.l. solutions (or well-posed in the class of continuous solutions), then (assuming that A, $\mathscr{D}(A)$ satisfy some mild conditions) it follows that $p_t(x, \cdot)$ defined by

$$p_t(x, A) = \mathbb{P}(X_t^x \in A) \tag{1.3}$$

is a transition probability function and any solution is a Markov process with p_t as its transition probability function (see e.g. Theorems IV.4.2 and IV.4.6 of [7]). This in turn gives us the associated semi-group (T_t) and its generator L. The generator L happens to be an extension of the operator A and thus A contains all the 'relevant information' about L as well as about X.

We extend this result and show that if the martingale problem is well-posed in the class of solutions that are continuous in probability, then (under suitable conditions on A, $\mathscr{D}(A)$) the function p_t defined by (1.3) is measurable.

In order to achieve our aim we give a Borel structure to the set of distributions of processes that are continuous in probability. Once we have done this, we can deduce that well-posedness in the class of solutions which are continuous in probability implies measurability of the associated transition probability function.

In §4, we give criterion for a measure to be invariant for the semigroup generated by a well-posed martingale problem. This is an improvement on several results on this theme (see [1,2,3,4,6,9,10]). In the last section, we give examples of operators (and their domains) satisfying the conditions of §3, and such that the corresponding martingale problems are well-posed in the class of solutions that are continuous in probability but for which no r.c.l.l. solution exists.

2. Preliminaries

We will denote by (E, d) a complete, separable metric space. A will denote an operator with domain $D(A) \subset C_b(E)$, the space of real-valued bounded continuous functions on E and with range contained in $M(E)$, the class of all real-valued Borel measurable functions on E. Let $B(E)$ denote the class of all bounded Borel measurable functions. For $C \subset B(E)$, we define the *bp*-closure of C to be the smallest subset of $B(E)$ containing C which is closed under bounded pointwise convergence of sequences of functions. $\mathscr{B}(E)$ will denote the Borel σ-field on E, $\mathscr{P}(E)$ will denote the space of probability measures on E. For a random variable Z taking values in E, $\mathscr{L}(Z)$ will denote the law of Z- i.e. the probability measure $\mathbb{P} \circ Z^{-1}$, if Z is defined on $(\Omega, \mathscr{F}, \mathbb{P})$. For a measurable Process(X_t) defined on $(\Omega, \mathscr{F}, \mathbb{P})$, let

$$^* \mathscr{F}_t^X = \sigma \left\{ X_u, \int_0^u h(X_s) \mathrm{d}s : u \le t, \, h \in C_b(E) \right\}.$$

Throughout this article, we will assume the following:

*Assumption A*1. There exists a $[0, \infty)$-valued measurable function Φ on E such that

$$|Af(x)| \le C_f \Phi(x) \quad \forall x \in E, \, f \in \mathscr{D}(A). \tag{2.1}$$

DEFINITION 2.1.

An E-valued process $(X_t)_{0 \le t < \infty}$ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a solution to the martingale problem for (A, μ) if

  (i) X is a measurable process with $\mathscr{L}(X_0) = \mu$,
  (ii) $\mathbb{E}_{\mathbb{P}} \left[\int_0^T \Phi(X_s) ds \right] < \infty$ for all $T < \infty$, and
 (iii) for every $f \in \mathscr{D}(A)$,

$$M_t^f = f(X_t) - \int_0^t Af(X_s) \, ds \tag{2.2}$$

is a $(^* \mathscr{F}_t^X)$-martingale. X will be called a solution to the A martingale problem if it is a solution to the (A, μ) martingale problem for some μ.

Let $\mathscr{W}$ be a class of E-valued processes. For example, we could consider $\mathscr{W}$ to be the class of E-valued processes with r.c.l.l. paths or $\mathscr{W}$ can be the class of solutions that are continuous in probability.

DEFINITION 2.2.

The martingale problem for A is said to be well-posed in the class $\mathscr{W}$ if for all $x \in E$, there exists a solution $X^x \in \mathscr{W}$ to the (A, δ_x) martingale problem and if $Y \in \mathscr{W}$ is any other solution to the (A, δ_x) martingale problem, then the finite dimensional distributions of X^x and Y are the same.

We begin with some observations on solutions to the A-martingale problem.

Theorem 2.1. *Let X (defined on some $(\Omega, \mathscr{F}, \mathbb{P})$) be a solution of the martingale problem for A. Suppose that $\mathscr{D}(A)$ is a determining class and further that*

$$t \longrightarrow \mathscr{L}(X_t) \text{ is continuous.} \tag{2.3}$$

Then $t \longrightarrow X_t$ is continuous in probability.

Proof. Let $f \in \mathscr{D}(A)$. The assumption (2.3) alongwith the fact that the martingale M^f (see eq. (2.2)) has a r.c.l.l. modification implies that M^f is continuous in probability. This in turn implies that the mapping $t \longrightarrow f(X_t)$ is continuous in probability. As a consequence, for $f, g \in \mathscr{D}(A)$,

$$(s,t) \longrightarrow \mathbb{E}_{\mathbb{P}}[f(X_s)g(X_t)] \text{ is continuous.} \tag{2.4}$$

Let $s_k \to s$. The assumption (2.3) implies that the family of distributions $\{\mathscr{L}(X_{s_k})\}$ is tight and so the family of distributions (on $E \times E$)

$$\{\mathscr{L}(X_{s_k}, X_s) : k \ge 1\} \text{ is tight.} \tag{2.5}$$

Since the class of functions $(x,y) \longrightarrow f(x)g(y)$, $f, g \in \mathscr{D}(A)$ constitutes a determining class, (2.4) and (2.5) together imply that

$$\mathscr{L}(X_{s_k}, X_s) \to \mathscr{L}(X_s, X_s). \tag{2.6}$$

Now for any $\varepsilon > 0$, $\mathbb{P}(d(X_s, X_s) \geq \varepsilon) = 0$. Thus in view of (2.6)

$$\limsup_{k \to \infty} \mathbb{P}(d(X_{s_k}, X_s) \geq \varepsilon) \leq 0$$

i.e.,

$$\mathbb{P}(d(X_{s_k}, X_s) \geq \varepsilon) \to 0.$$

This completes the proof. ∎

Remark 2.2. The proof given above contains the proof of the following: if for a process Y, the mapping $(s,t) \longrightarrow \mathscr{L}(Y_s, Y_t)$ is continuous, then Y is continuous in probability.

Remark 2.3. The assumption (2.3) can be replaced by

$$\{X_t : 0 \leq t \leq T\} \text{ is tight } \forall T < \infty.$$

COROLLARY 2.4.

Let X (defined on some $(\Omega, \mathscr{F}, \mathbb{P})$) be a solution of the martingale problem for A. Suppose that the domain $\mathscr{D}(A)$ of A is a convergence determining class on E. Then the process X is continuous in probability.

Proof. Since $f(X_t) - \int_0^t Af(X_s)\mathrm{d}s$ is a martingale for $f \in \mathscr{D}(A)$, it follows that the mapping $t \longrightarrow \mathbb{E}_\mathbb{P}[f(X_t)]$ is continuous. Since $\mathscr{D}(A)$ is a convergence determining class, this implies continuity of the mapping

$$t \longrightarrow \mathscr{L}(X_t).$$

Thus, by Theorem 2.1, $t \longrightarrow X_t$ is continuous in probability. ∎

3. Main result

We have seen in the previous section that under suitable conditions, all solutions to a martingale problem are continous in probability. Thus we now construct a Borel structure on the class of distributions of such processes.

For $m \geq 1$, E^m with the product topology is again a complete separable metric space. Let $\mathscr{P}(E^m)$ be equipped with the topology of weak convergence. Let $\mathscr{C}_m = C([0,\infty)^m, \mathscr{P}(E^m))$ be equipped with the topology of uniform convergence on compact subsets. Then $\mathscr{C}_m$ is a complete separable metric space. Let $\mathscr{S}_m$ be the set of $\mu^m = \mu^m(t_1, t_2, \ldots, t_m) \in \mathscr{C}_m$ satisfying

$$\int (\pi f)(x_1, x_2, \ldots, x_m) \mu^m(t_{\pi 1}, t_{\pi 2}, \ldots, t_{\pi m})(\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_m)$$

$$= \int f(x_1, x_2, \ldots, x_m) \mu^m(t_1, t_2, \ldots, t_m)(\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_m) \tag{3.1}$$

for all permutations π of $\{1, 2, \ldots, m\}$, for all $f \in C_b(E^m)$ where πf is defined by

$$\pi f(x_1, x_2, \ldots, x_m) = f(x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi m}).$$

It is easy to see that $\mathscr{S}_m$ is a closed subset of $\mathscr{C}_m$ and hence $\mathscr{S}_m$ is a complete separable metric space. Let $\mathscr{S}_\infty = \Pi_{m=1}^\infty \mathscr{S}_m$. Under the product topology, $\mathscr{S}_\infty$ is also a complete separable metric space. Elements of $\mathscr{S}_\infty$ will be denoted by $\mu = (\mu^1, \mu^2, \dots)$ with $\mu^k \in \mathscr{S}_k$. Let D denote the diagonal in E^2,

$$D = \{(x,x) \colon x \in E\}$$

and let

$$\mathscr{H} = \{\mu^2 \in \mathscr{S}_2 \colon \mu^2(t,t)(D) = 1 \quad \forall t \in [0,\infty)\}.$$

Since D is closed in E^2 and $\mu^2 \in \mathscr{S}_2$ is continuous, it follows that $\mathscr{H}$ is a closed subset of $\mathscr{S}_2$. Let

$$\mathscr{S}^* = \{\mu \in \mathscr{S}_\infty \colon \mu^m(t_1, \dots, t_m) \circ (h_m)^{-1} = \mu^{m-1}(t_1, \dots, t_{m-1}), \forall m > 1\},$$

where $h_m \colon E^m \longrightarrow E^{m-1}$ is the projection map defined by

$$h_m(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_{m-1}).$$

Let

$$\tilde{\mathscr{S}} = \{\mu \in \mathscr{S}^* \colon \mu^2 \in \mathscr{H}\}.$$

Then clearly $\tilde{\mathscr{S}}$ is also a complete separable metric space since it is a closed subspace of $\mathscr{S}_\infty$. Every element of $\tilde{\mathscr{S}}$ is a *consistent* family of finite dimensional distributions and hence by the Kolmogorov consistency theorem, given $\mu = (\mu^1, \mu^2, \dots) \in \tilde{\mathscr{S}}$, there exists a probability space $(\Omega^*, \mathscr{F}^*, P^*)$ and a stochastic process (X_t) on it such that for all $m \geq 1$,

$$\mathscr{L}(X_{t_1}, X_{t_2}, \dots, X_{t_m}) = \mu^m(t_1, t_2, \dots, t_m). \tag{3.2}$$

In view of Remark 2.2 and the fact that $\mu^2 \in \mathscr{H}$, the process X is continuous in probability. Conversely, given a E-valued process X that is continuous in probability, μ^m defined by (3.2) belongs to $\mathscr{S}_m$, μ^2 belongs to $\mathscr{H}$ and clearly $\{\mu^1, \mu^2, \dots\}$ is a consistent family and hence $\mu = (\mu^1, \mu^2, \dots) \in \tilde{\mathscr{S}}$. Thus, $\tilde{\mathscr{S}}$ can be identified with the class of distributions of E-valued processes that are continuous in probability.

Having given a topological structure to the class of (distributions of) processes that are continuous in probability, we now identify the class of (distributions of) solutions to the martingale problem for A and show that it is a Borel set. As in the corresponding result on solutions with r.c.l.l. paths [7], we assume that $A, \mathscr{D}(A)$ satisfy the following:

*Assumption A*2. There exists a countable set $\{f_n \colon n \geq 1\} \subset \mathscr{D}(A)$ such that

$$bp - \text{closure}\{(f_n, \Phi^{-1}Af_n) \colon n \geq 1\} \supset \{(f, \Phi^{-1}Af) \colon f \in \mathscr{D}(A)\}.$$

Let X be a process that is continuous in probability (on some $(\Omega, \mathscr{F}, \mathbb{P})$). Since every such process admits a measurable modification [5], we assume that X is measurable. Let $\mathscr{G}$ be a countable bp-dense subset of $C_b(E)$. Then X is a solution to the A martingale problem if and only if

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^N \phi(X_u) du\right] < \infty \quad \forall N \geq 1$$

and

$$\mathbb{E}_{\mathbb{P}}\left[g_1(X_{s_1})\dots g_k(X_{s_k})\left(f_m(X_t)-f_m(X_s)-\int_s^t (Af_m)(X_u)\mathrm{d}u\right)\right]=0$$

for all $s_1,s_2,\dots,s_k,s,t$ rationals with $s_i \le s \le t$, $g_i \in \mathscr{G}$, $1 \le i \le k$, $k \ge 1, m \ge 1$, where $\{f_j: j \ge 1\}$ are as in Assumption A2. Thus, a measurable process X is a solution to the A martingale problem if and only if its finite dimensional distributions $\mu = (\mu^1,\mu^2,\dots)$ defined by (3.2) belong to $\mathscr{M} \subset \tilde{\mathscr{S}}$ defined as follows: $\mathscr{M}$ is the set of $\mu = (\mu^1,\mu^2,\dots) \in \tilde{\mathscr{S}}$ satisfying

$$\int_0^N \langle\mu^1(s),\Phi\rangle\mathrm{d}s < \infty \quad \forall N \ge 1 \tag{3.3}$$

(here, $\langle F,\Gamma\rangle$ denotes $\int F\mathrm{d}\Gamma$) and

$$\langle\mu^{k+1}(s_1,s_2,\dots,s_k,t),G\otimes f_m\rangle - \langle\mu^{k+1}(s_1,s_2,\dots,s_k,s),G\otimes f_m\rangle$$
$$=\int_s^t \langle\mu^{k+1}(s_1,s_2,\dots,s_k,u),G\otimes Af_m\rangle\mathrm{d}u \tag{3.4}$$

for all $s_1,s_2,\dots s_k,s,t$ rationals with $s_i \le s \le t$, $g_i \in \mathscr{G}$, $1 \le i \le k$, $k \ge 1, m \ge 1$, where $\{f_j: j \ge 1\}$ are as in Assumption A2 and

$$G\otimes f_m(x_1,x_2,\dots,x_k,z)=g_1(x_1)g_2(x_2)\dots g_k(x_k)f_m(z).$$

Since $\mathscr{M}$ is defined via countably many conditions with each condition in turn involving measurable functions of $\mu = (\mu^1,\mu^2,\dots)$, it follows that $\mathscr{M}$ is a Borel subset of $\tilde{\mathscr{S}}$. Moreover, given $\mu = (\mu^1,\mu^2,\dots) \in \mathscr{M}$, as noted above there exists a process X such that its finite dimensional distributions are those given by $\mu = (\mu^1,\mu^2,\dots)$. Further, $\mu^2 \in \mathscr{H}$ and Remark 2.2 implies that this process is continuous in probability and can be assumed to be measurable. It follows that X is a solution to the A martingale problem. We have thus proved the following.

Theorem 3.1. *Suppose that $A,\mathscr{D}(A)$ satisfy Assumptions A1 and A2. Then $\mu = (\mu^1,\mu^2,\dots) \in \mathscr{M}$ if and only if there exists a process X that is* (i) *continuous in probability,* (ii) *the finite dimensional distributions of X are given by $\mu = (\mu^1,\mu^2,\dots)$ and* (iii) *X is a solution to the martingale problem for A.*

We are now ready to prove the measurability of p_t when the martingale problem for A is well-posed. We introduce the following:

Assumption A3. The martingale problem for (A,δ_x) is well-posed in the class of solutions that are continuous in probability for each $x \in E$.

Theorem 3.2. *Suppose that $A,\mathscr{D}(A)$ satisfy A1, A2 and A3. Let X^x denote a solution that is continuous in probability to the (A,δ_x) martingale problem. Let $p_t(x,B)$, $t \in [0,\infty)$, $x \in E$, $B \in \mathscr{B}(E)$ be defined by*

$$p_t(x,B)=P(X_t^x \in B). \tag{3.5}$$

Then for all $t \in [0,\infty)$, $B \in \mathscr{B}(E)$, $x \longrightarrow p_t(x,B)$ is Borel measurable.

Proof. Note that $F = \{\delta_x \colon x \in E\}$ is a Borel measurable subset of $\mathscr{P}(E)$ (indeed it is a closed subset) and the function $\theta(\delta_x) = x$ is a Borel measurable function on it (again this is a continuous function). Let $\psi_t \colon \mathscr{M} \longrightarrow \mathscr{P}(E)$ for $0 \leq t < \infty$ be defined by

$$\psi_t(\mu) = \mu^1(t), \quad \mu = (\mu^1, \mu^2, \dots) \in \mathscr{M}.$$

The functions ψ_t are continuous and hence measurable. Let $\mathscr{M}_0 = (\psi_0)^{-1}(F)$. It follows that $\mathscr{M}_0$ is a Borel subset of $\tilde{\mathscr{S}}$. Also, $\Psi = \theta \circ \psi_0$ is a measurable function from $\mathscr{M}_0$ into E.

In view of the Assumption A3, for a given $x \in E$, $\mathscr{M}$ has exactly one element $\mu = (\mu^1, \mu^2, \dots)$ such that

$$\mu^1(0) = \delta_x$$

and hence the function Ψ is one-to-one. Hence by Kurtowski's theorem (see e.g. Corollary I.3.3 of [11]) the function is bimeasurable, or it has a measurable inverse. Let us note that $\Psi^{-1}(x)$ denotes the finite dimensional distributions of X^x- the (unique in law) solution to (A, δ_x) martingale problem which is continuous in probability. The required conclusion follows by noting that

$$p_t(x, B) = \psi_t(\Psi^{-1}(x))(B). \qquad \blacksquare$$

*Assumption A*4. $\mathscr{D}(A)$ is convergence determining.

*Assumption A*5. The (A, δ_x) martingale problem is well-posed in the class of measurable processes for all $x \in E$.

Remark 3.3. Let us note that Assumptions A4 and A5 imply Assumption A3. This is because Assumption A4 implies that every solution to the A martingale problem is continuous in probability. Thus the conclusion of the above theorem remains valid with the same proof if instead we assume that $A, \mathscr{D}(A)$ satisfy A1, A2, A4 and A5.

Remark 3.4. Assume that Assumptions A1, A2 and A3 are true. Denote by

$$\mu_x = (\mu_x^1, \mu_x^2, \dots)$$

the finite dimensional distributions of the (unique in law) solution to the (A, δ_x) martingale problem that is continuous in probability. We have seen in the proof above that

$$x \longrightarrow \mu_x(= \Psi^{-1}(x))$$

is Borel measurable and hence for all $t_1, t_2, \dots, t_m, m \geq 1$

$$x \longrightarrow \mu_x^m(t_1, t_2, \dots, t_m) \text{ is Borel measurable.} \qquad (3.6)$$

The next step is to prove that $\{T_t \colon t \geq 0\}$ defined by

$$T_t f(x) = \int f(y) p_t(x, \mathrm{d}y) = \int f(y) \mu_x^1(t)(\mathrm{d}y) \qquad (3.7)$$

is a semigroup on the class of bounded Borel measurable functions f on E. For this, we need to consider the martingale problem with non-degenerate initial distributions. Note

that well-posedness for degenerate initials in the class of all solutions may not imply well-posedness for all initials. To proceed further, let us introduce the following notation:

$$\Phi_N^*(x) = \int_0^N \langle \mu_x^1(s), \Phi \rangle ds. \tag{3.8}$$

Then in view of Remark 3.4, it follows that Φ_N^* is a $[0, \infty)$-valued measurable function. The next lemma shows that the existence of solution to the martingale problem holds for a large class of initial distributions.

Let $\mathscr{P}_\Phi$ be the set of all measures $\lambda \in \mathscr{P}(E)$ such that

$$\langle \Phi_N^*, \lambda \rangle < \infty, \quad \forall N \geq 1. \tag{3.9}$$

Lemma 3.5. *Suppose that $A, \mathscr{D}(A)$ satisfy Assumptions A1, A2 and A3. Let $\lambda \in \mathscr{P}_\Phi$. Then $\nu = (\nu^1, \nu^2, \dots)$ defined by*

$$\langle \nu^m(t_1, t_2, \dots, t_m), g \rangle = \int \langle \mu_x^m(t_1, t_2, \dots, t_m), g \rangle d\lambda(x) \tag{3.10}$$

belongs to $\mathscr{M}$ with $\nu^1(0) = \lambda$. Hence there exists a solution to the martingale problem for (A, λ) (whose finite dimensional distributions are $\{\nu^m\}$).

Proof. It is easy to see that $\{\nu^m\}$ satisfy (3.4) since each $\{\mu_x^m\}$ satisfies the same. Further, condition (3.9) on λ alongwith the definition of Φ_N^* implies that ν^1 satisfies (3.3) and hence $\{\nu^m\}$ belongs to $\mathscr{M}$. Thus the corresponding process Y is a solution to the martingale problem for (A, λ). ∎

We need one more observation on martingale problems before we can state our result on (T_t) defined by (3.7).

Lemma 3.6. *Let a process X defined on $(\Omega, \mathscr{F}, \mathbb{P})$ be a solution to the (A, λ) martingale problem and let g be a $[0, M]$-valued measurable function on E (where $M < \infty$) such that $\langle \lambda, g \rangle = 1$. Let γ be defined by $d\gamma/d\lambda = g$. Let $\mathbb{Q}$ be defined by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = g(X_0).$$

Then, considered as a process on $(\Omega, \mathscr{F}, \mathbb{Q})$, X is a solution to the (A, γ) martingale problem.

Proof. Since g is bounded it follows that

$$\mathbb{E}_\mathbb{Q}\left[\int_0^N \Phi(X_u)du\right] \leq M\mathbb{E}_\mathbb{P}\left[\int_0^N \Phi(X_u)du\right]$$
$$< \infty.$$

Moreover, since $d\mathbb{Q}/d\mathbb{P}$ is $\sigma(X_0)$ measurable, it follows that $f(X_t) - \int_0^t Af(X_u)du$ is a martingale on $(\Omega, \mathscr{F}, \mathbb{Q})$ (as it is a martingale on $(\Omega, \mathscr{F}, \mathbb{P})$). The result follows upon noting that $\mathbb{Q} \circ (X_0)^{-1} = \gamma$. ∎

In addition to Assumption A3, we need to assume the following in order to show that $\{T_t\}$ is a semigroup.

Assumption A6. There exists a sequence $\{h_n : n \geq 1\}$ of $[0, \infty)$-valued Borel measurable functions on E such that for every $\lambda \in \mathscr{P}(E)$ satisfying

$$\langle h_n, \lambda \rangle < \infty \quad \forall n \geq 1, \tag{3.11}$$

any two solutions to the (A, λ) martingale problem that are continuous in probability have the same finite-dimensional distributions.

Thus, in order to verify that Assumption A6 holds in a given example, we can show that the uniqueness holds under finitely many (or even countably many) integrability condition(s). We are now in a position to prove the semigroup property of (T_t). In the course of the proof, we also get, with little extra work, the result that every solution to the martingale problem satisfies the Markov property. The Markov property can also be obtained by following arguments as in [8].

Theorem 3.7. *Suppose that $A, \mathscr{D}(A)$ satisfy Assumptions A1, A2, A3 and A6.*

(i) *The martingale problem for (A, λ) is well-posed in the class of solutions that are continuous in probability if and only if $\lambda \in \mathscr{P}_\Phi$. Further, the finite-dimensional laws of the solution Y that is continuous in probability are given by (3.10).*

(ii) *Let $\lambda \in \mathscr{P}_\Phi$. Let X be a solution to the (A, λ) martingale problem (defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$). Further, let X be continuous in probability. Then X is a Markov process and the associated semigroup $\{T_t : t \geq 0\}$ is defined by (3.7).*

Proof.

(i) Let $\lambda \in \mathscr{P}_\Phi$. We have seen in Lemma 3.5 that the (A, λ) martingale problem admits a solution X whose finite-dimensional distributions are given by (3.10). Let X be defined on $(\Omega, \mathscr{F}, \mathbb{P})$. This process X is continuous in probability. Let Y be another solution to the (A, λ) martingale problem defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ such that Y is continuous in probability. Define g on E by

$$g(x) = C \sum_{n=1}^{\infty} 2^{-n} \frac{1}{1 + h_n(x)},$$

where C is a constant that is chosen so that $\langle \lambda, g \rangle = 1$. Define probability measures γ, $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda} = g, \quad \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = g(X_0) \quad \text{and} \quad \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\tilde{\mathbb{P}}} = g(Y_0).$$

By Lemma 3.6, X on $(\Omega, \mathscr{F}, \mathbb{Q})$ and Y on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{Q}})$ are solutions to the (A, γ) martingale problem. Further, these processes are continuous in probability. By construction, γ satisfies (3.11) and hence by Assumption A6, the finite-dimensional distributions of X on $(\Omega, \mathscr{F}, \mathbb{Q})$ are the same as those of Y on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{Q}})$. This in turn implies that the finite-dimensional distributions of X on $(\Omega, \mathscr{F}, \mathbb{P})$ are the same as those of Y on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. This proves well-posedness of the martingale problem for (A, λ).
Conversely, let X be a solution of the (A, λ) martingale problem that is continuous in probability. This time define

$$g(x) = C \sum_{n=1}^{\infty} 2^{-n} \frac{1}{1 + \Phi_n^*(x)},$$

where C is a constant that is chosen so that $\langle \lambda, g \rangle = 1$. Define probability measures γ and $\mathbb{Q}$ by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda} = g, \quad \text{and} \quad \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = g(X_0).$$

By Lemma 3.6, X is a solution to the (A, γ) martingale problem under $\mathbb{Q}$ and X is continuous in $\mathbb{Q}$ probability. By the first part, we have that the regular conditional probability distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_m})$ given by $\sigma(X_0)$ is $\mu_{X_0}^m(t_1, t_2, \ldots, t_m)$. As a consequence

$$\mathbb{E}_{\mathbb{Q}} \left[\int_0^N \Phi(X_s) \mathrm{d}s \big| \sigma(X_0) \right] = \Phi_N^*(X_0). \tag{3.12}$$

Since $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is $\sigma(X_0)$ measurable, (3.12) implies that

$$\mathbb{E}_{\mathbb{P}} \left[\int_0^N \Phi(X_s) \mathrm{d}s \big| \sigma(X_0) \right] = \Phi_N^*(X_0)$$

and hence

$$\mathbb{E}_{\mathbb{P}} \left[\int_0^N \Phi(X_s) \mathrm{d}s \right] = \mathbb{E}_{\mathbb{P}} \left[\Phi_N^*(X_0) \right]$$
$$= \langle \Phi_N^*, \lambda \rangle.$$

Since X is a solution to the (A, λ) martingale problem, the LHS above is finite for all N and hence $\lambda \in \mathscr{P}_\Phi$.

(ii) Let X be a solution to the (A, λ) martingale problem that is continuous in probability (defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$). Fix $m \geq 1$ and $0 \leq u_1 < u_2 < \cdots < u_m \leq s$ and $h_1, h_2 \ldots h_m$ bounded positive continuous functions. Define a probabiltiy measure $\mathbb{Q}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = C h_1(X_{u_1}) h_2(X_{u_2}) \ldots h_m(X_{u_m}),$$

where the constant C is chosen such that $\mathbb{Q}$ is a probability measure. Define Y by

$$Y_t = X_{s+t}, \quad t \geq 0.$$

Then using $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ which is bounded (say by M), we get

$$\mathbb{E}_{\mathbb{Q}} \left[\int_0^T \Phi(Y_u) \mathrm{d}u \right] = \mathbb{E}_{\mathbb{Q}} \left[\int_s^{T+s} \Phi(X_u) \mathrm{d}u \right]$$
$$\leq M \mathbb{E}_{\mathbb{P}} \left[\int_s^{T+s} \Phi(X_u) \mathrm{d}u \right] < \infty. \tag{3.13}$$

Further, it can be shown that Y is a solution to the (A, γ) martingale problem where $\gamma = \mathbb{Q} \circ [Y(0)]^{-1}$. Of course, Y is continuous in probability. Hence, by part (i) above we get that $\gamma \in \mathscr{P}_\Phi$ and that the finite-dimensional distributions are given by (3.10) (with λ replaced by γ). Thus, for $g_1, \ldots, g_k \in C_b(E)$ and $0 \leq s_1 < \cdots < s_k$,

$$\mathbb{E}_{\mathbb{Q}} \left[g_1(Y_{s_1}) \ldots g_k(Y_{s_k}) \right] = \int \langle \mu_x^k(s_1, \ldots, s_k), g_1 \otimes \cdots \otimes g_k \rangle \mathrm{d}\gamma(x)$$
$$= \mathbb{E}_{\mathbb{Q}} \left[\langle \mu_{Y_0}^k(s_1, \ldots, s_k), g_1 \otimes \cdots \otimes g_k \rangle \right]$$

and so (using $k = 1$, $s_1 = t$ and $g_1 = g$) we can conclude that

$$\mathbb{E}_{\mathbb{P}}\left[Ch_1(X_{u_1})h_2(X_{u_2})\ldots h_m(X_{u_m})g(X_{s+t})\right]$$

$$= \mathbb{E}_{\mathbb{Q}}\left[g(Y_t)\right] = \mathbb{E}_{\mathbb{Q}}\left[\langle \mu^1_{Y_0}(t), g \rangle\right]$$

$$= \mathbb{E}_{\mathbb{P}}\left[Ch_1(X_{u_1})h_2(X_{u_2})\ldots h_m(X_{u_m})\langle \mu^1_{X_s}(t), g \rangle\right]$$

for all $0 \leq u_1 < u_2 < \cdots < u_m \leq s$ and $h_1, h_2 \ldots h_m$ bounded positive continuous functions, $m \geq 1$. As a consequence,

$$\mathbb{E}_{\mathbb{P}}\left[g(X_{s+t})\big|\sigma(X_u: 0 \leq u \leq s)\right] = \langle \mu^1_{X_s}(t), g \rangle = (T_t g)(X_s).$$

This completes the proof. ∎

4. Criterion for an invariant measure

Several papers gave criterion for a measure to be invariant for the semigroup (T_t) arising from a well-posed martingale problem [1,2,3,4,6,9,10]. These papers assumed different sets of conditions on $(A, D(A))$. It was shown that existence of solution for each degenerate initial and

$$\int (Af)d\lambda = 0 \quad \forall f \in \mathscr{D}(A)$$

gives existence of a stationary solution of the martingale problem for (A, λ). In addition, if the martingale problem is well-posed and there is a semigroup (T_t) associated with it, it follows that λ is an invariant measure for (T_t).

Well-posedness of the martingale problem in the class of r.c.l.l. solutions is sufficient for the existence of the semigroup (T_t) (see Theorem 4.4.6 of [7]).

In the light of the results obtained in the previous section, we can improve on this criterion for invariant measure.

We introduce another condition on A and Φ (appearing in Assumption A1).

Assumption A7. Φ and Af, for every $f \in \mathscr{D}(A)$, are continuous.

Lemma 4.1. *Suppose that $A, \mathscr{D}(A)$ satisfy Assumptions A1, A2, A3 and A7. Then A satisfies the positive maximum principle, i.e. if $f \in \mathscr{D}(A)$ and $z \in E$ are such that $f(z) \geq 0$ and $f(z) \geq f(x)$ for all $x \in E$, then*

$$Af(z) \leq 0.$$

Proof. Let X be a solution to (A, δ_z) martingale problem defined on $(\Omega, \mathscr{F}, \mathbb{P})$ that is continuous in probability. Let $\mathscr{F}_t = {}^*\mathscr{F}^X_t$ and

$$M_t = f(X_t) - \int_0^t Af(X_u)du.$$

Then $(M_t, \mathscr{F}_t)$ is a martingale. Let $\sigma_t, 0 \leq t < \infty$ be the increasing family of $(\mathscr{F}_t)$ stopping times defined by

$$\sigma_t = \inf\left\{s \geq 0: \int_0^s (1 + \Phi(X_u))du \geq t\right\}.$$

Note that $\sigma_t \leq t$ for all t. Since $\mathbb{E}_{\mathbb{P}}[\int_0^s \Phi(X_u) \mathrm{d}u] < \infty$, it follows that σ_t increases to ∞ a.s..

Let $N_t = M_{\sigma_t}$, $Y_t = X_{\sigma_t}$ and $\mathscr{G}_t = \mathscr{F}_{\sigma_t}$. Then, it follows that $(N_t, \mathscr{G}_t)$ is a local martingale. Moreover, $t \longrightarrow \sigma_t$ is continuous and hence Y is also continuous in probability. Using change of variable, it is easy to see that

$$N_t = f(Y_t) - \int_0^t \frac{Af(Y_r)}{1 + \Phi(Y_r)} \mathrm{d}r.$$

Since $Af(x) \leq C_f \Phi(x)$, it follows that N is bounded and hence is a martingale. Since f has a maximum at z and

$$\mathbb{E}_{\mathbb{P}} \left[f(Y_t) - f(z) - \int_0^t \frac{Af(Y_r)}{1 + \Phi(Y_r)} \mathrm{d}r \right] = 0,$$

it follows that (using Fubini's theorem)

$$\int_0^t \mathbb{E}_{\mathbb{P}} \left[\frac{Af(Y_r)}{1 + \Phi(Y_r)} \right] \mathrm{d}r \leq 0 \quad \forall t > 0. \tag{4.1}$$

Since Y is continuous in probability and $Af(x) \leq C_f \Phi(x)$, it follows that

$$r \longrightarrow \mathbb{E}_{\mathbb{P}} \left[\frac{Af(Y_r)}{1 + \Phi(Y_r)} \right]$$

is continuous. Now dividing the LHS in (4.1) by t and taking limit as $t \to 0$ we get

$$\frac{Af(z)}{1 + \Phi(z)} \leq 0.$$

Since $\Phi(z) \geq 0$ this completes the proof. ∎

Here is yet another assumption on $A, \mathscr{D}(A)$.

*Assumption A*8. $\mathscr{D}(A)$ is an algebra that contains constants and separates points in E.

Theorem 4.2. *Suppose that $A, \mathscr{D}(A)$ satisfy Assumptions A1, A2, A3, A6, A7 and A8. Let (T_t) be the semigroup associated with $(A, \mathscr{D}(A))$ by Theorem 3.7.*
If $\lambda \in \mathscr{P}(E)$ is such that $\int \Phi \mathrm{d}\lambda < \infty$ and

$$\int (Af)(x) \mathrm{d}\lambda(x) = 0 \quad \forall f \in \mathscr{D}(A), \tag{4.2}$$

then λ is an invariant measure for the semigroup (T_t) and the solution to the (A, λ) martingale problem that is continuous in probability is a stationary process.

Proof. In view of Lemma 4.1 and the assumptions made in the statement of this theorem, the proof of Theorem 3.1 in [3] gives the existence of a stationary solution to the (A, λ) martingale problem. Since the solution (say X) is stationary, the mapping $t \longrightarrow \mathscr{L}(X_t)$ is continuous (it is a constant) and hence by Theorem 2.1, X is continuous in probability. Now, Theorem 3.7 implies that λ is an invariant measure for (T_t). ∎

Remark 4.3. The criterion for invariant measure given above is true even if Assumption A7 above is not true but instead one assumes that the operator A satisfies Assumptions A9, A10 and A11 given below. This is helpful, e.g., when Af is allowed to be a discontinuous function (see [4,9]).

*Assumption A*9. A satisfies the positive maximum principle.

*Assumption A*10. There exists a complete separable metric space U, an operator $\hat{A}\colon \mathscr{D}(A) \to C(E \times U)$ and a transition function η from $(E, \mathscr{B}(E))$ into $(U, \mathscr{B}(U))$ such that

$$(Af)(x) = \int_U \hat{A}f(x,u)\eta(x,\mathrm{d}u). \tag{4.3}$$

*Assumption A*11. There exists $\hat{\Phi} \in C(E \times U)$ such that for all $f \in \mathscr{D}(A)$, there exists $C_f < \infty$ satisfying

$$|\hat{A}f(x,u)| \leq C_f \hat{\Phi}(x,u) \quad \forall x,u \in E \times U, \tag{4.4}$$

$$\Phi(x) = \int_U \hat{\Phi}(x,u)\eta(x,\mathrm{d}u) < \infty. \tag{4.5}$$

Under these conditions, existence of a stationary solution to the (A, λ) martingale problem was proven in [4]. Rest of the argument is as in the proof of the above theorem.

5. Example

We give two examples of processes that are continuous in probability and which arise as solutions of well-posed martingale problems but such that they do not admit any r.c.l.l. modification. The results of the previous section, however, are applicable.

Example 5.1. Let $E = [0,1)$. Let $\mathscr{D}(A)$ be the class of functions f that are restrictions of some periodic function $g \in C_b^2(\mathbb{R})$ with period 1. Further for $f \in \mathscr{D}(A)$ define Af by $Af = \frac{1}{2}f''$. Then A and $\mathscr{D}(A)$ satisfy the
conditions of Theorems 3.2 and 3.7.

It follows easily that if W is a one-dimensional standard Brownian motion then $X_t = W_t$ (mod 1) is a solution to the martingale problem for A. Moreover, for any other solution Y of the martingale problem, it is easy to check that Y behaves like a Brownian motion as long as it does not hit the boundary. Now, uniqueness can be shown using localisation arguments as in Theorem 6.6.1 of [13].

Note that almost every path of the unique solution X is neither r.c.l.l. nor l.c.r.l. However the set of discontinuity points of X is contained in the set

$$\{t\colon W_t \text{ is an integer}\}.$$

This implies that X is continuous in probability.

Example 5.2. Let $E = (0, \infty)$ and let μ be a probability measure on E with $\mu\{(0,a)\} = 0$ for some $a > 0$. Let $\mathscr{D}(A)$ be defined by

$$\mathscr{D}(A) = \left\{ f \in C_b^2(E)\colon \lim_{x \to 0} f(x) = \int f \, \mathrm{d}\mu \right\}.$$

For $f \in \mathscr{D}(A)$ define Af by $Af = \frac{1}{2}f''$. Once again, A and $\mathscr{D}(A)$ satisfy the conditions of Theorems 3.2 and 3.7.

The uniqueness of solution for the martingale problem for A can also be shown using localisation arguments as in Theorem 6.6.1 of [13]. To construct the unique solution for the (A, δ_x) martingale problem we can proceed as follows.

Let $\{W^{z,i}: i \geq 0\}$ be independent one-dimensional standard Brownian motions starting at z. Define

$$\tau^{z,i} = \inf\{t > 0: W^{z,i} = 0\}.$$

Note that

$$\tau^{z,i} < \infty \text{ a.s. for every } z,i. \tag{5.1}$$

Let $U_1, U_2, \ldots$ be i.i.d. random variables with common distribution μ and which are independent of all $\{W^{z,i}: i \geq 1\}$. Define

$$X_t^x = \begin{cases} W_t^{x,0}, & \text{for } t < \tau^{x,0}, \\ W_t^{U_i,i}, & \text{for } \tau^{U_{i-1},i-1} \leq t < \tau^{U_i,i}; \quad i \geq 1. \end{cases}$$

Then it is easily checked that X^x is a solution of the martingale problem for A starting at x and which is also continuous in probability. Then (5.1) and the fact that $0 \notin E$ together imply that almost every path of X^x is not left continuous.

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